

DETECTING BIFURCATION VALUES AT INFINITY OF REAL POLYNOMIALS

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ABSTRACT. We present a new approach for estimating the set of bifurcation values at infinity. This yields a significant shrinking of the number of coefficients in the recent algorithm introduced by Jelonek and Kurdyka for reaching critical values at infinity by rational arcs.

1. INTRODUCTION

The *bifurcation locus* of a polynomial mapping $f: \mathbb{K}^n \rightarrow \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $n \geq p$ is the smallest subset $B(f) \subset \mathbb{K}^p$ such that f is a locally trivial C^∞ -fibration over $\mathbb{K}^p \setminus B(f)$. It is well known that $B(f) = f(\text{Sing} f) \cup \mathcal{B}_\infty(f)$, where $\mathcal{B}_\infty(f)$ denotes the set of *bifurcation values at infinity* (Definition 2.1). A simple example is $f(x, y) = x + x^2y$, where we have $\text{Sing} f = \emptyset$ and $\mathcal{B}_\infty(f) = \{0\}$.

While the set of critical values $f(\text{Sing} f)$ is relatively well understood, the other set $\mathcal{B}_\infty(f)$ is still mysterious. In case $p = 1$ the bifurcation set $B(f)$ is finite, as proved by Thom [Th], see also [Ph, Ve]. However, one can precisely detect the bifurcation set $B(f)$ only in case $p = 1$ and $n = 2$ by using several types of (equivalent) tests, see [Su], [HL], [Ti2], [Dur], [Ti3] over \mathbb{C} and [TZ], [CP] over \mathbb{R} .

For $p = 1$ and more than two variables one can only estimate $B(f)$ by “reasonably good” supersets $A(f) \supset B(f)$ of the form $A(f) = f(\text{Sing} f) \cup A_\infty(f)$, where $A_\infty(f)$ is a finite set which depends on the choice of some regularity condition at infinity: *tameness* [Br1], *Malgrange regularity* [Pa], ρ -*regularity* [NZ], [Ti1], t -*regularity* [ST], [Ti2].

In case $p > 1$ it is known that $\mathcal{B}_\infty(f)$ is contained in a one codimensional semi-algebraic subset of \mathbb{R}^p , or an algebraic subset of \mathbb{C}^p , respectively, as proved by Kurdyka, Orro and Simon [KOS]. Sharper such subsets have been obtained in [DRT].

This paper addresses the problem of estimating the bifurcation locus at infinity $\mathcal{B}_\infty(f)$ and is motivated by the recent algorithm presented by Jelonek and Kurdyka [JK2] for finding the set of values $K_\infty(f)$ for which the *Malgrange condition* at infinity fails (Definition 2.7). Their algorithm applies to a space $AV(f)$ of rational arcs in \mathbb{R}^n , the coefficients of which are solutions of a certain set of equations.

We present here the construction of a different space of rational arcs $\text{Arc}(f)$ to which we attach the same set of equations for the coefficients, and thus the same algorithm can be run.

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Our main results, Theorem 3.4 together with Theorems 2.5 and 2.8, show that the resulting subset of asymptotic arcs $\text{Arc}_\infty(f)$ detects a certain set of values that includes $\mathcal{B}_\infty(f)$ and is included in $K_\infty(f)$. Consequently, the Jelonek-Kurdyka algorithm applied to our $\text{Arc}(f)$ will produce a (smaller) set of values including $\mathcal{B}_\infty(f)$ in considerably shorter time since the number of coefficients of the rational arcs is drastically reduced, namely:

$$\dim \text{Arc}(f) = n(1 + d^n) \quad \text{versus} \quad \dim AV(f) = n(2 + d(d+1)^n(d^n + 2)^{n-1}).$$

Our result is also relevant for optimisation and complexity problems since bifurcation values at infinity appear for instance in the optimization of real polynomials, e.g. [HP2], [Sa].

2. REGULARITY CONDITIONS AT INFINITY

2.1. Bifurcation values at infinity. We start by recalling the basic definitions after [Ti2], [Ti3], [DRT], eventually adapting them to any $n \geq p \geq 1$.

Definition 2.1. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. We say that $t_0 \in \mathbb{R}^p$ is a *typical value* of f if there exists a disk $D \subset \mathbb{R}^p$ centered at t_0 such that the restriction $f|_D: f^{-1}(D) \rightarrow D$ is a locally trivial C^∞ -fibration. Otherwise we say that t_0 is an *bifurcation value* (or atypical value). We denote by $B(f)$ the *bifurcation locus*, i.e. set of bifurcation values of f .

We say that f is *topologically trivial at infinity* at $t_0 \in \mathbb{R}^p$ if there exists a compact set $K \subset \mathbb{R}^n$ and a disk $D \subset \mathbb{R}^p$ centered at t_0 such that the restriction $f|_D: f^{-1}(D) \setminus K \rightarrow D$ is a locally trivial C^∞ -fibration. Otherwise we say that t_0 is a *bifurcation value at infinity* of f . We denote by $\mathcal{B}_\infty(f)$ the set of bifurcation values at infinity of f .

As we have claimed in the Introduction, there is no general characterisation¹ of the bifurcation locus besides the setting $n = 2$ and $p = 1$. In case $n > 2$ one uses regularity conditions at infinity in order to control the topological triviality. We work here with the ρ -regularity and with the Malgrange-Kuo-Rabier condition.

2.2. The ρ -regularity. We adapt the definition of ρ -regularity from [Ti2] to Euclidean spheres centered at any point of \mathbb{R}^n . This allows us to define a new set, denoted here by $S_\infty(f)$, which produces a sharper estimation of $B(f)$.

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and let $\rho_a: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be the Euclidian distance function to a , i.e. $\rho_a(x) = (x_1 - a_1)^2 + \dots + (x_n - a_n)^2$.

Definition 2.2 (Milnor set at infinity). Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. We fix $a \in \mathbb{R}^n$. We call *Milnor set of (f, ρ_a)* the critical set of the mapping $(f, \rho_a): \mathbb{R}^n \rightarrow \mathbb{R}^{p+1}$ and denote it by $M_a(f)$.

In case $p = 1$ we need the following statement, which was noticed in [HP1] (see also [Dut, Lemma 2.2]). We provide a proof and use some details of it in the proof of Theorem 3.4.

¹Still, one can treat particular cases when “singularities at infinity” are isolated in a certain sense, e.g. [ST], [Pa], [Ti3], [TT].

Lemma 2.3. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial mapping. There exists an open dense subset $\Omega_f \subset \mathbb{R}^n$ such that, for every $a \in \Omega_f$, $M_a(f) \setminus \text{Sing}f$ is either a non-singular curve or it is empty.*

Proof. We claim that the following semi-algebraic set:

$$(1) \quad Z := \{(x, a) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in M_a(f) \setminus \text{Sing}f\}.$$

is a smooth $(n+1)$ -dimensional manifold. For some fixed $(x_0, a_0) \in Z$, there is $1 \leq i \leq n$ such that $\frac{\partial f}{\partial x_i}(x_0) \neq 0$ and moreover, since $\text{Sing}f$ is closed, there exists an open set $U \subset \mathbb{R}^n$ such that $\frac{\partial f}{\partial x_i}(x) \neq 0, \forall x \in U$. For $1 \leq j \leq n, j \neq i$, we set:

$$(2) \quad m_j(x, a) := \frac{\partial f}{\partial x_i}(x)(x_j - a_j) - \frac{\partial f}{\partial x_j}(x)(x_i - a_i).$$

We have $Z \cap (U \times \mathbb{R}^n) = \{(x, a) \in U \times \mathbb{R}^n \mid m_j(x, a) = 0; 1 \leq j \leq n, j \neq i\}$. Let $\varphi: U \times \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the mapping $\varphi = (m_1, \dots, m_n)$, where m_i is missing. By its definition we have $\varphi^{-1}(0) = Z \cap (U \times \mathbb{R}^n)$. Let us notice that the rank of the gradient matrix $D\varphi$ is $n-1$ at any point of $U \times \mathbb{R}^n$. Indeed, the minor $(\frac{\partial m_l}{\partial a_k})_{1 \leq l, k \leq n}^{l, k \neq i}(x, a)$ is a diagonal matrix of which the entries on the diagonal are all equal to $-\frac{\partial f}{\partial x_i}(x)$, hence non-zero. This shows that Z is a manifold of dimension $n+1$.

We next consider the projection $\tau: Z \rightarrow \mathbb{R}^n, \tau(x, a) = a$. Thus, $\tau^{-1}(a) = (M_a(f) \setminus \text{Sing}f) \times \{a\}$. By Sard's Theorem, we conclude that, for almost all $a \in \mathbb{R}^n$, $\tau^{-1}(a) = (M_a(f) \setminus \text{Sing}f) \times \{a\} \cong (M_a(f) \setminus \text{Sing}f)$ is a smooth curve or that it is an empty set. \square

Definition 2.4 (ρ_a -regularity at infinity). Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. Let $a \in \mathbb{R}^n$. We call:

$$(3) \quad S_a(f) := \{t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset M_a(f), \lim_{j \rightarrow \infty} \|x_j\| = \infty \text{ and } \lim_{j \rightarrow \infty} f(x_j) = t_0\}.$$

the set of asymptotic ρ_a -nonregular values. If $t_0 \notin S_a(f)$ we say that t_0 is ρ_a -regular at infinity. We set $\mathcal{S}_\infty(f) := \bigcap_{a \in \mathbb{R}^n} S_a(f)$.

The above condition is a ‘‘Milnor type’’ condition that controls the transversality of the fibres of f to the spheres centered at $a \in \mathbb{R}^n$.²

We derive the following result about $\mathcal{S}_\infty(f)$ from [DRT, Theorem 5.7, Proposition 6.4] where it was stated for $S_0(f)$:

Theorem 2.5. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n \geq p \geq 1$. Then $\mathcal{B}_\infty(f) \subset \mathcal{S}_\infty(f)$.*

Moreover:

- (a) *The sets $\mathcal{S}_\infty(f)$ and $f(\text{Sing}f) \cup \mathcal{S}_\infty(f)$ are closed sets.*
- (b) *For any $a \in \mathbb{R}^n$, $S_a(f)$ and $f(\text{Sing}f) \cup S_a(f)$ are semi-algebraic sets of dimension $\leq p-1$.*

²For complex polynomial functions, transversality to large spheres was used in [Br1, p.229], in [NZ] where it is called *M-tameness*, later in [ST], [Dur] etc. In the real setting, it was used in [Ti2], later in [HP1] etc.

Proof. To prove the inclusion $\mathcal{B}_\infty(f) \subset \mathcal{S}_\infty(f)$, let $t_0 \in \mathbb{R}^p \setminus \mathcal{S}_\infty(f)$. It follows that there exists $a \in \mathbb{R}^n$ such that $t_0 \notin S_a(f)$. By [DRT, Prop. 6.4] we obtain that f is topologically trivial at infinity at $t_0 \in \mathbb{R}^p$, after Definition 2.1, where the later is an open set (as shown at (a) below). A completely similar reasoning shows the inclusion $B(f) \subset f(\text{Sing} f) \cup \mathcal{S}_\infty(f)$, thus ending the proof of the first part of our statement.

(a). The proof of [DRT, Theorem 5.7(a), p. 337] yields that $S_0(f)$ and $S_0(f) \cup f(\text{Sing} f)$ are closed sets. The same proof holds true when the point 0 is replaced by any other point $a \in \mathbb{R}^n$. This implies that $\mathcal{S}_\infty(f) = \bigcap_{a \in \mathbb{R}^n} S_a(f)$ and $\bigcap_{a \in \mathbb{R}^n} A_{\rho_a}(f)$ are closed sets.

The claim (b) was proved as [DRT, Theorem 5.7(b)] for $S_0(f)$ and the same proof holds for any $a \in \mathbb{R}^n$. \square

REMARK 2.6. Since $M_a(f)$ is semi-algebraic, for any value $c \in S_a(f)$ there exist paths $\phi :]0, \varepsilon[\rightarrow M_a(f) \subset \mathbb{R}^n$ such that $\lim_{t \rightarrow 0} \|\phi(t)\| = \infty$ and $\lim_{t \rightarrow 0} f(\phi(t)) = c$. This follows from the Curve Selection Lemma at infinity, as remarked in [DRT] and [CDTT].

2.3. Relation with the Malgrange-Kuo-Rabier condition. Jelonek and Kurdyka [JK1] gave an estimation for $B(f)$ by using the notion of asymptotic critical values of f . This is based on the Malgrange-Kuo-Rabier regularity at infinity. We have shown in [Ti1] and [DRT] that this condition implies the ρ -regularity at infinity, where ρ denotes the Euclidean distance, but that they are not the same and therefore the associated sets of “critical values at infinity” may be different (see Example 2.9).

Definition 2.7. [Ra] Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n \geq p$. Let $Df(x)$ be the Jacobian matrix of f at x . Let

$$(4) \quad K_\infty(f) := \{t \in \mathbb{R}^p \mid \exists \{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n, \lim_{j \rightarrow \infty} \|x_j\| = \infty, \\ \lim_{j \rightarrow \infty} f(x_j) = t \text{ and } \lim_{j \rightarrow \infty} \|x_j\| \nu(Df(x_j)) = 0\},$$

where $\nu(A) := \inf_{\|\varphi\|=1} \|A^*(\varphi)\|$, for a linear mapping $A : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and its adjoint $A^* : (\mathbb{R}^p)^* \rightarrow (\mathbb{R}^n)^*$. We call $K_\infty(f)$ the set of *asymptotic critical values of f* .

Note that in case $p = 1$ the last limit in (4) amounts to $\lim_{j \rightarrow \infty} \|x_j\| \|\text{grad} f(x_j)\| = 0$ thus the above definition recovers the definition of *Malgrange non-regular values at infinity* (cf [Pa], [ST], [Ti1] etc).

The next key result shows the inclusion³ $S_a(f) \subset K_\infty(f)$ for any $a \in \mathbb{R}^n$ (which may be strict, see Example 2.9). Moreover, it shows not only that all the values of $S_a(f)$ may be detected by paths ϕ like in the above Remark 2.6, but that the same paths ϕ verify the conditions (4) in the definition of $K_\infty(f)$. This will be a key ingredient in the proof of Theorem 3.4.

Theorem 2.8. *Let $f = (f_1, \dots, f_p) : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a polynomial mapping, where $n > p$. Let $\phi :]0, \varepsilon[\rightarrow M_a(f) \subset \mathbb{R}^n$ be an analytic path such that $\lim_{t \rightarrow 0} \|\phi(t)\| = \infty$ and $\lim_{t \rightarrow 0} f(\phi(t)) = c$. Then $\lim_{t \rightarrow 0} \|\phi(t)\| \nu(Df(\phi(t))) = 0$.*

In particular $S_a(f) \subset K_\infty(f)$ and $\mathcal{S}_\infty(f) \subset K_\infty(f)$.

³The particular inclusion $S_0(f) \subset K_\infty(f)$ was proved in [CDTT, Proposition 2.4].

Proof. Let $c = (c_1, \dots, c_p) \in S_a(f)$. By the definition of $M_a(f)$, we have $\phi(t) \in M_a(f)$ if and only if $\text{rank } D(f, \rho_a)(\phi(t)) < p + 1$. It follows that there exist real coefficients $\lambda(t), b_1(t), \dots, b_p(t)$ which are actually analytic functions of t and not all equal to zero for the same t , such that:

$$(5) \quad \lambda(t) \cdot (\phi_1(t) - a_1, \dots, \phi_n(t) - a_n) = b_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \dots + b_p(t) \frac{\partial f_p}{\partial x}(\phi(t)),$$

where $\frac{\partial f_i}{\partial x}(\phi(t)) := \left(\frac{\partial f_i}{\partial x_1}(\phi(t)), \dots, \frac{\partial f_i}{\partial x_n}(\phi(t)) \right)$ for $i = 1, \dots, p$.

Let us denote $\hat{\lambda}(t) := \frac{\lambda(t)}{\|b(t)\|}$ and $\hat{b}(t) := \frac{b(t)}{\|b(t)\|}$, where $b(t) = (b_1(t), \dots, b_p(t))$. Since $b(t) \neq 0 \forall t \in]0, \varepsilon[$ for small enough ε , the equality (5) writes:

$$(6) \quad \sum_{i=1}^p \hat{b}_i(t) \frac{\partial f_i}{\partial x}(\phi(t)) = \hat{\lambda}(t)(\phi(t) - a).$$

From this we obtain:

$$(7) \quad \sum_{i=1}^p \hat{b}_i(t) \frac{d}{dt} f_i(\phi(t)) = \left\langle \sum_{i=1}^p \hat{b}_i(t) \frac{\partial f_i}{\partial x}(\phi(t)), \phi'(t) \right\rangle = \frac{1}{2} \hat{\lambda}(t) \frac{d}{dt} \|\phi(t) - a\|^2.$$

We shall denote by ord_t the order at 0 of some analytic parametrisation. The condition $\lim_{t \rightarrow 0} f_i(\phi(t)) = c_i$ implies that $\text{ord}_t \left(\frac{d}{dt} f_i(\phi(t)) \right) \geq 0$, $i = 1, \dots, p$, and the condition $\|\hat{b}(t)\| = 1$ implies that the order of the first sum in (7) is also non-negative. We may therefore derive from (7):

$$(8) \quad 0 \leq \text{ord}_t \left(\hat{\lambda}(t) \frac{d}{dt} \|\phi(t) - a\|^2 \right) < \text{ord}_t \left(\hat{\lambda}(t) \|\phi(t) - a\|^2 \right).$$

By taking norms in (5) and multiplying by $\|\phi(t) - a\|$ we get:

$$(9) \quad \text{ord}_t \left(\|\phi(t) - a\| \left\| \hat{b}_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \dots + \hat{b}_p(t) \frac{\partial f_p}{\partial x}(\phi(t)) \right\| \right) = \text{ord}_t \left(|\hat{\lambda}(t)| \|\phi(t) - a\|^2 \right),$$

which is positive by (8). This implies:

$$\lim_{t \rightarrow 0} \|\phi(t) - a\| \left\| \hat{b}_1(t) \frac{\partial f_1}{\partial x}(\phi(t)) + \dots + \hat{b}_p(t) \frac{\partial f_p}{\partial x}(\phi(t)) \right\| = 0,$$

which, in turn, implies $\lim_{t \rightarrow 0} \|\phi(t) - a\| \nu(dF(\phi(t))) = 0$. Since $\text{ord}_t \|\phi(t) - a\| = \text{ord}_t \|\phi(t) - a\| < 0$, we get $\lim_{t \rightarrow 0} \|\phi(t)\| \nu(dF(\phi(t))) = 0$, which shows that $c \in K_\infty(F)$.

In order to show the claimed inclusions $S_a(f) \subset K_\infty(f)$ and $\mathcal{S}_\infty(f) \subset K_\infty(f)$ we use Remark 2.6 and the above proof. \square

The inclusion $\mathcal{S}_\infty(f) \subset K_\infty(f)$ may be strict, as showed by the next example.

EXAMPLE 2.9. [PZ] The polynomials $f_{nq} : \mathbb{K}^3 \rightarrow \mathbb{K}$, $f_{nq}(x, y, z) := x - 3x^{2n+1}y^{2q} + 2x^{3n+1}y^{3q} + yz$, where $n, q \in \mathbb{N} \setminus \{0\}$. We have $S_0(f_{nq}) = \emptyset$. For $\mathbb{K} = \mathbb{C}$, it is shown in [PZ] that f_{nq} satisfies Malgrange's condition for any $t \in \mathbb{C}$ if and only if $n \leq q$. One can check that the same holds for $\mathbb{K} = \mathbb{R}$. For $n > q$ we therefore get $S_0(f_{nq}) \subsetneq K_\infty(f_{nq}) \neq \emptyset$.

2.4. Questions and Conjecture. We have seen above that the inclusions $B(f) \subset S_a(f) \cup f(\text{Sing} f) \subset K_\infty(f) \cup f(\text{Sing} f)$ hold for any $a \in \mathbb{R}^n$. On other hand, one can have $S_a(f) \neq S_b(f)$, as shown in the following example for which we skip the computations:

EXAMPLE 2.10. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = y(x^2 y^2 + 3xy + 3)$. Then $S_{(0,0)}(f) = \emptyset$ and $S_{(0,1)}(f) \neq \emptyset$.

In particular, for the polynomial $g(x, y) := f(x, y-1)$ we have that $\emptyset \neq S_0(g) \not\subset \mathcal{B}_\infty(g) = \emptyset$ which is a real counterexample to a conjecture in [NZ, p. 686, point 3]. A new conjecture can be stated as below.

These facts support the following natural questions:

- (i) Is there a minimal set $S_a(f)$ in the collection $\{S_b(f), b \in \mathbb{R}^n\}$?
- (ii) If (i) is true, does the minimality hold for some open dense subset $a \in \mathbb{R}^n$?

The following conjecture seems also natural:

Conjecture 2.11. $\mathcal{B}_\infty(f) = \mathcal{S}_\infty(f)$.

3. DETECTING BIFURCATION VALUES AT INFINITY BY PARAMETRIZED CURVES

3.1. Bounding the set of asymptotic critical values $\mathcal{S}_\infty(f)$. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function of degree $d \geq 2$. If $t_0 \in \mathcal{S}_\infty(f)$ then, by Proposition 2.3 and Definition 2.4, there exists an open dense set of points $a \in \mathbb{R}^n$ such that $M_a(f)$ is a real affine algebraic set of dimension 1 and there exists a real asymptotic branch $\Gamma \subset M_a(f)$ such that $\lim_{x \in \Gamma, \|x\| \rightarrow \infty} f(x) = t_0$. Therefore such branches Γ detect all the values in $\mathcal{S}_\infty(f)$.

Moreover, $\#S_a(f)$ is precisely the number of finite values taken by f when restricted to branches at infinity of $M_a(f)$. It follows that $\#\mathcal{S}_\infty(f)$ is majorated by the number of branches at infinity.

This interpretation yields the same bound for the number of bifurcation values at infinity $\#\mathcal{B}_\infty(f)$ as the bound found by Jelonek and Kurdyka for $\#K_\infty(f)$.

Proposition 3.1. [JK1, Corollary 1.2]

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function of degree $d \geq 2$. The number $\#\mathcal{S}_\infty(f)$ of asymptotical critical values is majorated by $d^{n-1} - 1$.

Proof. We may assume that $M_a(f)$ is not empty for $a \in \Omega_f$. By the proof of Lemma 2.3, $M_a(f)$ is a real curve which is a local complete intersection defined by $n - 1$ equations as in (2). Let us consider the closure $\overline{M_a(f)}$ in $\mathbb{P}_\mathbb{R}^n$.

The complexification $M_a(f)^\mathbb{C}$ of $M_a(f)$ is defined locally as the complex solutions of the same system of equations, hence it is a complex curve of degree equal to d^{n-1} . Let us denote by $\overline{M_a(f)^\mathbb{C}}$ its closure in $\mathbb{P}_\mathbb{C}^n$, and notice that this is a complex curve of the same degree d^{n-1} .

The hyperplane at infinity $H^\infty := \mathbb{P}_\mathbb{C}^n \setminus \mathbb{C}^n$ intersects $\overline{M_a(f)^\mathbb{C}}$ at finitely many points and by Bezout's theorem this number is bounded by the degree d^{n-1} . If $B(f) \setminus f(\text{Sing} f) \neq \emptyset$ then the restriction of $f_\mathbb{C}$ to $M_a(f)^\mathbb{C}$ cannot be bounded, since there is at least one branch on which the holomorphic function f is non-constant. Hence we can get at most $d^{n-1} - 1$ finite values as limits of f . \square

3.2. Branches at infinity of the Milnor set and asymptotic parametrisations. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a polynomial function of degree $d \geq 2$. If $t_0 \in \mathcal{S}_\infty(f)$ then, by Proposition 2.3 and Definition 2.4, there exists an open dense set of points $a \in \mathbb{R}^n$ such that $M_a(f)$ is a real affine algebraic set of dimension 1 and moreover, there exists a real asymptotic branch $\Gamma \subset M_a(f)$ such that $\lim_{x \in \Gamma, \|x\| \rightarrow \infty} f(x) = t_0$.

Let us describe briefly following the ideas of [JK2] how to produce real parametrisations of the branches at infinity of $M_a(f)$. Let $M_a(f)^\mathbb{C}$ denote its complexification defined by the same equations. We recall that this is a locally complete intersection and has degree $D := d^{n-1}$. One may construct Puiseux parametrisations for the branches at infinity $\Gamma_\mathbb{C}$ of $M_a(f)^\mathbb{C}$ like in [JK2, Lemma 3.1] $\gamma_\mathbb{C}(t) = \sum_{-\infty \leq i \leq D} c_i t^i$, where $t \in \mathbb{C}$ and $|t| > R$ for some radius $R \gg 1$. The bound to the right is $D = d^{n-1}$.

Since each real branch at infinity Γ of $M_a(f)$ is contained in some complex branch of $\Gamma_\mathbb{C}$ of $M_a(f)^\mathbb{C}$ then, following Milnor's procedure [Mi, pag. 28-29], one may further construct⁴ a real parametrisation $\gamma(t) = \sum_{-\infty \leq i \leq D} a_i t^i$ of Γ with the same bound $D = d^{n-1}$ to the right, where $t \in \mathbb{R}$ with $|t| > R$ and the same radius $R \gg 1$.

Next we truncate the parametrisation $\gamma(t)$ at the left to the bound $-D'$, where $D' := (d-1)D$ and d is the degree of f . This truncation $\hat{\gamma}(t) = \sum_{-(d-1)D \leq i \leq D} a_i t^i$ does not verify anymore the equations of $M_a(f)$. Nevertheless we can show the following:

Proposition 3.2. *The above defined truncation $\hat{\gamma}(t)$ has the following properties:*

- (a) $\lim_{t \rightarrow \infty} \|\hat{\gamma}(t)\| = \lim_{t \rightarrow \infty} \|\gamma(t)\| = \infty$.
- (b) $\lim_{t \rightarrow \infty} f(\hat{\gamma}(t)) = \lim_{t \rightarrow \infty} f(\gamma(t)) = t_0$.
- (c) $\lim_{t \rightarrow \infty} \frac{\partial f}{\partial x_i}(\hat{\gamma}(t)) = \lim_{t \rightarrow \infty} \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$, for any i .
- (d) $\lim_{t \rightarrow \infty} x_j \frac{\partial f}{\partial x_i}(\hat{\gamma}(t)) = \lim_{t \rightarrow \infty} x_j \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$, for any i, j .

Proof. (a) is obvious from the definitions.

(b) follows from [JK2, Lemma 3.3] since the degree of f is d .

(c) and (d). We claim that the limits involving $\gamma(t)$ are zero. If this is true, then the same will follow for the truncation $\hat{\gamma}(t)$ by using [JK2, Lemma 3.3] since the degrees of $\frac{\partial f}{\partial x_i}$ and of $x_j \frac{\partial f}{\partial x_i}$ are both $\leq d$.

In order to prove our claim, we use our key result Theorem 2.8 for $p = 1$, as follows. Since in this case $\nu(Df(x)) = \|\text{grad} f(x)\|$, the equality $\lim_{t \rightarrow 0} \|\gamma(t)\| \|(\frac{\partial f}{\partial x_1}(\gamma(t)), \dots, \frac{\partial f}{\partial x_n}(\gamma(t)))\| = 0$ provided by Theorem 2.8 clearly implies $\lim_{t \rightarrow \infty} x_j \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$ and $\lim_{t \rightarrow \infty} \frac{\partial f}{\partial x_i}(\gamma(t)) = 0$.

This completes our proof. \square

3.3. The arc space. Jelonek and Kurdyka [JK2] found an algorithm for reaching the values of $K_\infty(f)$ by parametrized curves with bounded expansion. They construct a finitely dimensional space of such rational arcs (cf [JK2, Definition 6.9]). Based on our framework we construct a new space of rational arcs of considerably lower dimension. We shall see that this detects at least the bifurcation set $\mathcal{S}_\infty(f)$ without necessarily detecting all the values in $K_\infty(f)$.

⁴this part of our construction is different from the procedure given in [JK2, Lemma 3.2].

We consider here the following space of arcs associated to the real polynomial $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree d :

$$(10) \quad \text{Arc}(f) = \left\{ \xi(t) = \sum_{-(d-1)d^{n-1} \leq k \leq d^{n-1}} a_k t^k, a_k \in \mathbb{R}^n \right\}$$

which is isomorphic to $\mathbb{R}^{n(1+d^n)}$.

REMARK 3.3. The dimension of the arc space $AV(f)$ constructed in [JK2] is $n(2 + d(d + 1)^n(d^n + 2)^{n-1})$, while ours is $\dim \text{Arc}(f) = n(1 + d^n)$.

In a similar manner as in [JK2, Definition 6.10], we define the *asymptotic variety of arcs*, $\text{Arc}_\infty(f) \subset \text{Arc}(f)$, as the algebraic subset of the rational arcs $\xi(t) \in \text{Arc}(f)$ such that:

- (a') $\sum_{k>0} \sum_{j=1}^n a_{kj}^2 = 1$, where $a_k = (a_{k1}, \dots, a_{kn})$.
- (b') $f(\xi(t)) = b_0 + \sum_{k=1}^\infty b_k t^{-k}$, where $b_0, b_k \in \mathbb{R}$.
- (c') $\frac{\partial f}{\partial x_i}(\xi(t)) = \sum_{k=1}^\infty c_{ik} t^{-k}$, for any i , where $c_{ik} \in \mathbb{R}$.
- (d') $x_j \frac{\partial f}{\partial x_i}(\xi(t)) = \sum_{k=1}^\infty d_{ijk} t^{-k}$, for any i, j , where $d_{ijk} \in \mathbb{R}$.

The conditions (a')–(d') for ξ are equivalent with the corresponding properties (a)–(d) applied to ξ insted of $\hat{\gamma}$. For instance the first equivalence follows by renormalising the coefficients.

Let $b_0 : \text{Arc}_\infty(f) \rightarrow \mathbb{R}$, $b_0(\xi(t)) = \lim_{t \rightarrow \infty} f(\xi(t))$.

Theorem 3.4. $\mathcal{S}_\infty(f) \subset b_0(\text{Arc}_\infty(f)) \subset K_\infty(f)$.

Proof. The inclusion $b_0(\text{Arc}_\infty(f)) \subset K_\infty(f)$ is a direct consequence of the definitions of $\text{Arc}_\infty(f)$ and $K_\infty(f)$ since properties (a'), (b') and (d') characterise the values $b_0 \in K_\infty(f)$.

Let us show the first inclusion. If $b_0 \in \mathcal{S}_\infty(f)$ then there is some $a \in \mathbb{R}^n$ and there exists a path $\gamma(t) \in M_a(f)$, such that $\lim_{t \rightarrow \infty} f(\gamma(t)) = b_0$. Then Theorem 2.8 shows that γ verifies the conditions (a)–(d) of Proposition 3.2. Moreover, by the same Proposition 3.2 one has that the truncation $\hat{\gamma}$ defined at §3.2 verifies the same properties, hence the equivalent conditions (a')–(d') too. This ends our proof. \square

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